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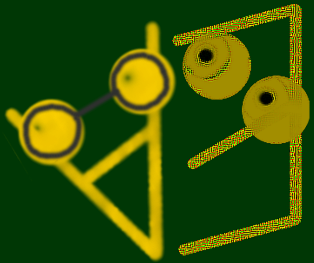
Mathematical Foundations for Computer Science

The Language of Mathematics



The most important thing I want to say is:

DON'T FREAK OUT ABOUT THIS!

We will release more information every day to help you out. We want to see everyone succeed and this question is no exception.

Re-iterating from yesterday: This question exists because we want you to get to know the TAs and learn how to ask good questions!

Here's some new information:

- The order of the office hours is strict. It will not be at all useful to see TAs out of order, because you won't know what questions to ask. Please let other people sign up for slots.
- We expect everyone to get full credit on this question. We will be more helpful as time goes on. Please don't panic.
- Melody's blog is supposed to be amusing. Don't take things on there (other than the blog post itself) too seriously. (I hear some of you were going to make the meatloaf!)
- Try to have fun with it! We know it's different, and weird, and frustrating, but you will be able to finish. We promise.
- We will keep on adding more office hours. Don't worry. You will see everyone you want to see.
- We're experimenting with the amount of time appointments should be. Clearly, 5 minutes isn't enough. In the future, we will make appointments longer. Then, if that's still not long enough, we'll do it again.

In this course, you will be learning a language just like English or `Python`.

Communicating using mathematics really does require that you learn the language, and it comes with its own idiosyncracies.

Over the semester, we'll build up words, sentences, paragraphs, and arguments.

Today is all about "words" and "sentences".

Some Basic Grammar

Here's some "sentences":

- $2 + 2 = 5$
- This is the song that never ends. . .
- This statement is false.
- Akjsdf?

Which of these sentences are "grammatically" correct? You should think of the question "does it make sense to talk about this sentence?"

Here's some "sentences":

- $2 + 2 = 5$ This sentence is false, but certainly we can talk about it.
- This is the song that never ends... What does "..." mean? There's something iffy about an "infinite" sentence.
- This statement is false. Can we really talk about a self-referential "statement"? What is its truth value?
- Akjsdf? This just plain makes no sense.

Definition

Mathematical Statement We say a sentence is a **mathematical statement** precisely when:

- it has a truth value, and
- it is "well-formed" (no "...", no gibberish)

Ironically, my definition of "mathematical statement" is itself not a mathematical statement.

This is because defining this formally is overly complicated and unimportant right now.

Consider the following:

> *"Roger, the orange elephant, has tusks only if he has toenails and he has toenails but not tusks. Also, Roger, the orange elephant, is orange."*

This is all fine, but, WTF does it mean?

The language of mathematics is going to allow us to (1) be more precise than English, (2) be more concise than English, and (3) figure out what a statement means more quickly.

"Roger, the orange elephant, has tusks only if he has toenails and he has toenails but not tusks. Also, Roger, the orange elephant, is orange."

Our language has to support defining properties or statements. To "translate" this sentence, we're going to need a couple ideas:

- IsOrange(x) := "x is orange".
- IsElephant(x) := "x is an elephant".
- HasTusks(x) := "x has tusks".
- HasToenails(x) := "x has toenails".

Before we start "translating"... what sorts of values do these "functions" return? Well, if we give them reasonable values like Roger, then they are either T (true) or F (false).

What does HasToenails(47) mean?

This is gibberish! It would be like asking "Does the sky eat vegetables?" or something else silly like that...

Our First "Translation"

"Roger, the orange elephant, has tusks only if he has toenails and he has toenails but not tusks. Also, Roger, the orange elephant, is orange."

- IsOrange(x) := "x is orange".
- IsElephant(x) := "x is an elephant".
- HasTusks(x) := "x has tusks".
- HasToenails(x) := "x has toenails".

IsOrange(Roger) **and** IsElephant(Roger) **and** (HasTusks(Roger) **only if** HasToenails(Roger)) **and** (HasToenails(Roger) **and not** HasTusks(Roger)) **and** IsOrange(Roger) **and** IsElephant(Roger) **and** IsOrange(Roger).

The statement "IsOrange(Roger) and IsElephant(Roger)" appears twice! Let's make another definition to make the sentence easier:

Let p be the statement "IsOrange(Roger) and IsElephant(Roger)".

> *"Roger, the orange elephant, has tusks only if he has toenails and he has toenails but not tusks. Also, Roger, the orange elephant, is orange."*

Let p be the statement "IsOrange(Roger) and IsElephant(Roger)".

- IsOrange(x) := "x is orange".
- IsElephant(x) := "x is an elephant".
- HasTusks(x) := "x has tusks".
- HasToenails(x) := "x has toenails".

p **and** (HasTusks(Roger) **only if** HasToenails(Roger)) **and** (HasToenails(Roger) **and not** HasTusks(Roger)) **and** p **and** IsOrange(Roger).

Looking at this sentence, we see that we're asserting "p and p" as part of it. This is the first situation where the power of logic really shines. The orange words are all "connectives" that combine various statements together. Let's investigate them...

This one is the most straight-forward. If we have statements p and q, we say:

- If p is T and q is T, then p and q is T.
- If p is T and q is F, then p and q is F.
- If p is F and q is T, then p and q is F.
- If p is F and q is F, then p and q is F.

This sort of definition of a logical connective is so common, it has a name and a shorthand. We call this a truth table, and we generally write them down like this:

p	q	p and q
T	T	T
T	F	F
F	T	F
F	F	F

Since "and" is such a common word in our language, it has its own symbol "$\wedge$". So, $p \wedge q$ means "p and q".

Disjunction (or)

Another very common connective is "or" which we write as "$\vee$".

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The biggest gotcha with mathematical or is that T $\vee$ T is T. In other words, "p or q" means "at least one of p and q is true".

Not!

Connectives don't have to be between two statements! Not is very common, and it works how you'd expect:

p	$\neg p$
T	F
F	T

Don't Forget Roger!

Let's introduce our new symbology to Roger's life...

Let p be the statement "IsOrange(Roger) and IsElephant(Roger)".

- IsOrange(x) := "x is orange".
- IsElephant(x) := "x is an elephant".
- HasTusks(x) := "x has tusks".
- HasToenails(x) := "x has toenails".

$p \wedge$ (HasTusks(Roger) **only if** HasToenails(Roger)) $\wedge$
(HasToenails(Roger) $\wedge$ ($\neg$HasTusks(Roger))) $\wedge\, p \wedge$ IsOrange(Roger).

The only thing left is that pesky "if". Let's fix that too!

Let's concern ourselves with statements like "If p, then q."

How Implication Works

"If p, then q." is a **promise** that whenever p is true, we immediately know q is true. We can figure out the truth value of "if p, then q" by asking the question "In this situation, has the promise been broken?"

Example (Implication)

If it is raining, then I have my umbrella.

First Question: It's not raining, and I don't bring my umbrella. Have I broken the promise?
Second Question: It's not raining, and I bring my umbrella. Have I broken the promise?
In both cases, the pre-requisite to my promise isn't met. So, I haven't in either case. In fact, the only case in which I've lied to you is when it's raining, but I don't have my umbrella.

Implication ("if")

p	q	$p \implies q$
T	T	T
T	F	F
F	T	T
F	F	T

The notation for "if p, then q" is $p \implies q$.

In Roger's sentence, we use the phrase "p only if q". What does it mean?

"I am a Pokemon master only if I have collected all 151 Pokemon"

We're guaranteeing that if I am a Pokemon master, I have all 151 pokemon. So, this is just

"If I am a pokemon master, I have all 151 pokemon".

$p \wedge (\text{HasTusks}(\text{Roger}) \implies \text{HasToenails}(\text{Roger})) \wedge (\text{HasToenails}(\text{Roger}) \wedge (\neg\text{HasTusks}(\text{Roger}))) \wedge p \wedge \text{IsOrange}(\text{Roger})$.

I'm going to re-arrange this sentence a little bit (You will prove that this is okay on Friday):

$p \wedge p \wedge \text{IsOrange}(\text{Roger}) \wedge (\text{HasTusks}(\text{Roger}) \implies \text{HasToenails}(\text{Roger})) \wedge (\text{HasToenails}(\text{Roger}) \wedge (\neg\text{HasTusks}(\text{Roger})))$.

A lot of the time, we want to make changes to sentences like this one. When we do, we often want to say "they have the same truth value".

We say "p is equivalent to q" or "p if and only if q" or "p iff q" or "$p \iff q$" when p and q always have the same truth value.

p	q	$p \iff q$
T	T	T
T	F	F
F	T	F
F	F	T

We can often use known equivalences to prove new ones! For instance, if we know that $p \iff q$, then we can replace p's with q's in sentences!

$$p \land p \land \text{IsOrange}(\text{Roger}) \land (\text{HasTusks}(\text{Roger}) \implies \text{HasToenails}(\text{Roger})) \land (\text{HasToenails}(\text{Roger}) \land (\neg \text{HasTusks}(\text{Roger}))).$$

We know $p \land p \iff p$. So, we can replace it:

$$p \land \text{IsOrange}(\text{Roger}) \land (\text{HasTusks}(\text{Roger}) \implies \text{HasToenails}(\text{Roger})) \land (\text{HasToenails}(\text{Roger}) \land (\neg \text{HasTusks}(\text{Roger}))).$$

Can we simplify this further? Think about what implication means.

We said before that for $p \implies q$, the only case in which I'm lying is when p is true, but q isn't. "$\text{HasTusks}(\text{Roger}) \implies \text{HasToenails}(\text{Roger})$" is actually asserting "$\neg \text{HasTusks}(\text{Roger}) \lor \text{HasToenails}(\text{Roger})$".

We know from the other part of the sentence that both of these are true. So, it seems Roger is in fact an orange elephant with toenails, but no tusks!

Introducing Roger's Friends

Now that we understand Roger's sentence, he's decided he's lonely. He would like us to talk about all of his friends.

Our current symbology doesn't allow us to express ideas like these:

> *Every elephant has tusks or is Roger.*
> *At least one elephant has toenails and is Roger.*

So, we introduce quantifiers:

$$\forall, \exists$$

$$\forall x.\, x \leq 0 \vee x > 0$$

$$\exists x.\, x \neq 0$$

But what is x?

In general, there's some reasonable set of values that we're quantifying over. It could be the integers, the "shapes", "people". Anything reasonable.

But we MUST specify what it is. The set we are quantifying over is called the domain of individuals.

$$\forall x.\, x \leq 0 \vee x > 0$$

Perhaps here the domain of individuals is the integers.

$$\exists x.\, x \neq 0$$

Maybe here, the domain of individuals is $\{0, 1, 2\}$.

People Examples

We let the domain of individuals be the set of people for every quantifier on this slide.

- Let CMU(x) be "x lives in a CMU dorm".
- Let Freshman(x) be "x is a freshman".
- Let Student(x) be "x is a student".

True or False:

$$\forall x. \text{Freshman}(x)$$

It's false. To prove that a $\forall$ statement is false, all we need to do is find one object in the domain of individuals that it's false for.

Importantly: we must actually specify a particular object. It is not enough to just say "one exists"!

So: This statement is false, because Adam is a person who is not a freshman.

We let the domain of individuals be the set of people for every quantifier on this slide.

- Let CMU(x) be "x lives in a CMU dorm".
- Let Freshman(x) be "x is a freshman".
- Let Student(x) be "x is a student".

True or False:

$$\forall x.\ (\text{Student}(x) \wedge \text{Freshman}(x)) \implies \text{CMU}(x)$$

This one is also false. Take a freshman at Pitt. She is a student and a freshman, but she doesn't live in a CMU dorm.

We let the domain of individuals be the set of people for every quantifier on this slide.

- Let CMU(x) be "x lives in a CMU dorm".
- Let Freshman(x) be "x is a freshman".
- Let Student(x) be "x is a student".

True or False:

$$\exists x.\ (\text{CMU}(x) \land \neg\text{Freshman}(x)) \implies \text{Student}(x)$$

The President of the United States is a person who does not live in a CMU dorm (which means the first part of the implication is false) and is not a student (which means the second part of the implication is false). So, we've found someone that makes the implication True.

Let the domain of individuals be the set of fruits on this slide.
Let Purple(x) mean "x is a purple fruit".

Consider the statement: $\forall x.\ \text{Purple}(x)$.

What is $\neg\forall x.\ \text{Purple}(x)$?

One possibility is $\exists x.\ \text{Purple}(x)$. Is this correct?

It isn't! Think back to the truth table for negation. It must always be the case that either a statement p or its negation $\neg p$ is true. In this case, in a world where the only fruit is an orange, neither statement is true!

The correct answer is $\exists x.\ \neg\text{Purple}(x)$. (or in words, "at least one fruit is not purple").

In general, the procedure to negate quantifiers, is to "push" the negation inside. Every time a negation "passes through" a quantifier, it switches.

Let the domain of individuals be people again.
Let $\mathsf{D}(x)$ be "x is a dreamer".
Consider the statement: "$\mathsf{D}(\mathsf{I}) \implies \exists x.\,(\mathsf{D}(x) \wedge x \neq \mathsf{I})$"
What is the negation?
The trick here is to get rid of the implication as quickly as possible. We saw earlier that $p \implies q \iff \neg p \vee q$.

$$\neg(\mathsf{D}(\mathsf{I}) \implies \exists x.\,(\mathsf{D}(x) \wedge x \neq \mathsf{I})) \iff \neg(\neg\mathsf{D}(\mathsf{I}) \vee \exists x.\,(\mathsf{D}(x) \wedge x \neq \mathsf{I}))$$

How do we continue? To be continued on Friday!

Sets Education

$\vee$

Sets All Folks!