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Mathematical Foundations for Computer Science

Sets

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Administrivia

- Meeting with TAs and me outside of office hours
- HW 1 is due tonight at 11:59pm.
- HW 2 comes out tonight!
- HW 2 has a “checkpoint”

Many of the homeworks will have "checkpoints". HW 2 has a "checkpoint" due this Friday.

Checkpoints shouldn't be scary. All they require is that you write yourself some notes about a particular homework problem (usually the first one on the assignment) and bring them to workshop.

Why are we doing this? Well... workshops are significantly less helpful if you haven't thought about the material at all outside of lecture. In particular, it really sucks to be in a group where one of the group members isn't prepared.

What if I ignore it? We will note it and–while you can still participate in the workshop–we will consider you to have "not attended". This is relevant, because, remember at the end of the semester if you have "missed" at most three workshops, then your lowest exam score can be replaced by your final score.

Last week we started learning a new language! But what use is a language if there's nothing to talk about?

Sets are the "things" we talk about when using the Mathematical Language.

Today, we will build up a basic working knowledge of sets.

Amazingly, we can build everything we'd ever want to talk about out of sets.

It would be incredibly tedious, but we could define numbers, addition, functions, ... all out of sets.

In fact, we've done this for you! Everything in {Setty} is made out of sets. We've given you an implementation of our mathematical language.

You should take advantage of this. When you are solving problems, use {Setty} to check small examples or your understanding.

Let's start with some not so great definitions of some common sets:

- $\mathbb{N}$ is the set of Natural Numbers; that is $\mathbb{N} = \{0, 1, 2, \dots\}$.
- $\mathbb{Z}$ is the set of Integers; that is $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$.
- $\mathbb{R}$ is the set of Real Numbers (e.g. $1, -17, \frac{32}{48}, \pi$)
- $\mathbb{Q}$ is the set of Rational Numbers (e.g. $\frac{1}{2}, 2, -1, \frac{-7}{99}$)
- $[n]$ is the set $\{1, 2, \dots, n\}$ when $n \in \mathbb{N}$.

And one more (incredibly) important set:

- $\{\} = \varnothing$ is the empty set; it is the only set with no elements in it.

Example (Sets)

(1) $\{1\}$

(2) $\{1, 3, 2\}$

(3) $\{\triangle, 1\}$

(4) $\{1, 2, 4, 8, \dots, 256\}$

There are a few notational issues with these definitions. Any thoughts?

Consider these sets:

- $\{1, 2, 3, 2, 1\}$
- $\{1, 1, 2, 2, 3\}$
- $\{1, 2, 3\}$

These are all sets. They just all happen to be the same set. The take-away here is sets ignore order and duplicates.

Also, let's look at $\{1, 2, 4, 8, \ldots, 256\}$ again. Any issues here?

The "..." isn't really well defined, even though we know what it means here. What about if we want something more complicated?

Definition (Set Comprehensions)

When we write

$$S := \{x \mid \mathsf{P}(x)\}$$

we are defining the set S as the "set of all x's where $\mathsf{P}(x)$ is true".

$$T := \{x \in S \mid \mathsf{P}(x)\}$$

we are defining T as the set of "those elements of S for which $\mathsf{P}(x)$ is true".

Example (Set Comprehensions)

"All the real numbers less than 1"

$$\{x \in \mathbb{R} \mid x < 1\}$$

"All the odd powers of two"

$$\{x \in \mathbb{N} \mid (\exists (n \in \mathbb{N}).\ x = 2^n) \wedge n \text{ is odd}\} = \{1\}$$

Now that we know how to construct sets, let's talk about the basic queries and operations.

Fundamentally, there is only one question we can ask about a set. Everything else is built on top of this question:

Definition (Set Membership)

Question: Is x a member of S? ($x \in S$)
Question: Is x **not** a member of S? ($x \notin S$)

Example (Set Membership)

Let $S := \{1,2,3\}$.

$1 \in S$? **Yes!**
$x \in S$? **This doesn't make sense. What is x?**
$\triangle \in S$? **No!**
$\varnothing \in S$? **This does actually make sense. It's false here though.**

Can we have sets inside other sets? Sure, why not?

Example (Sets in sets)

Let $S := \{\varnothing, \{1,2,3\}, 1\}$.

$1 \in S$? **Yes!**
$2 \in S$? **No! It's an element of an element of this set!**
$\{\} \in S$? **Yup! Remember, $\{\} = \varnothing$**

Consider the set $R := \{x \mid x \notin x\}$.
First, can we give an English description of this set? R is the set of all sets that don't contain themselves.

Is $R \in R$?

Suppose $R \in R$. By definition of R, $R \notin R$. This doesn't make any sense.

Suppose $R \notin R$. By definition of R, $R \in R$! This also doesn't make sense.

But those are the only two possibilities! What this set demonstrates is a **fundamental problem with "native set theory"**.

Luckily, it's difficult to run into this, and we can ignore it for this course.

What does it mean for two sets to be "equal"? Since the only question we can ask is "$x \in S$?", two sets are the same if they always agree:

Definition (Set Equality)

Let S, T be sets. We say $S = T$ ("S equals T") iff

$$\forall x.\, x \in S \iff x \in T$$

We have a similar idea for "$\leq$".

Definition (Subset)

Let S, T be sets. We say $S \subseteq T$ ("S is a **subset** of T") iff

$$\forall (x \in S).\, x \in T$$

Example (Subset)

Let $A := \{1,2,3\}$, $B := \{3,4,5\}$, $C := \{3,4\}$.
$\varnothing \subseteq A$? **Yes! The empty set is empty; A has all 0 elements in it!**
$A \subseteq B$? **No. B is missing 1.**
$C \subseteq B$? **Yes!**

Let S, T be sets.

Definition (Set Union)

$$S \cup T := \{x \mid x \in S \vee x \in T\}$$

Definition (Set Intersection)

$$S \cap T := \{x \mid x \in S \wedge x \in T\}$$

Definition (Set Difference)

$$S \setminus T := \{x \mid x \in S \wedge x \notin T\}$$

Example (Set Operations)

Let $A := \{1,2,3\}$, $B := \{4,5,6\}$, $C := \{3,4\}$.
How can we make these sets: $[6]$, $\{3\}$, $\{1,2\}$, $\{1,3\}$?

- $[6] = A \cup B = A \cup B \cup C$
- $\{3\} = A \cap C = A \setminus \{1,2\}$
- $\{1,2\} = A \setminus (A \cap C)$
- $\{1,3\} = A \setminus \{2\}$

Sometimes, we want to be able to union or intersect a bunch of things all at once. Let $S = \{s_1, s_2, \dots\}$ be a set.

Definition (Set Unary Union)

$$\bigcup S := s_1 \cup s_2 \cup \cdots$$

Definition (Set Unary Intersection)

$$\bigcap S := s_1 \cap s_2 \cap \cdots$$

Example (Unary Set Operations)

Question: What is $\bigcup\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$?
Answer: $\varnothing \cup \{\varnothing\} \cup \{\varnothing, \{\varnothing\}\} = \{\varnothing, \{\varnothing\}\}$

Question: What is $\bigcap\{\{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$?
Answer: $\{\varnothing\} \cap \{\varnothing, \{\varnothing\}\} = \{\varnothing\}$

Thinking back to the set operations we have, there's a very clear correspondance between several of these operations and the logical connectives we've learned. In particular: $\cup \sim \vee$, $\cap \sim \wedge$.

Our main proof strategy when trying to show that two sets are equal is going to be to drop to the definitions via logic.

(BTW, did you notice that $\neg$ is conspicuously missing? Back to that in a bit.)

Let's try out a sets proof.

Question

Let S,T,U be sets. Prove that $S\cup(T\cup U)=(S\cup T)\cup U$.

Proof

We want to prove that $S\cup(T\cup U)=(S\cup T)\cup U$. So, consider $S\cup(T\cup U)$:

$$\begin{aligned}
S\cup(T\cup U) &= \{x \mid x\in S\cup(T\cup U)\} && \text{[Set Comprehension]}\\
&= \{x \mid x\in S\vee x\in T\cup U\} && \text{[Definition of } \cup\text{]}\\
&= \{x \mid x\in S\vee(x\in T\vee x\in U)\} && \text{[Definition of } \cup\text{]}\\
&= \{x \mid (x\in S\vee x\in T)\vee x\in U\} && \text{[Associativity of } \vee\text{]}\\
&= \{x \mid x\in S\cup T\vee x\in U\} && \text{[Definition of } \cup\text{]}\\
&= \{x \mid x\in(S\cup T)\cup U\} && \text{[Definition of } \cup\text{]}\\
&= (S\cup T)\cup U && \text{[Set Comprehension]}
\end{aligned}$$

It follows that $S\cup(T\cup U)=(S\cup T)\cup U$

Let $S := \{1, 2, 3\}$. What is $\{x \mid x \notin S\}$?

$\{x \mid x \notin S\}$

This **thing** doesn't make any sense! There's no "restriction" on what goes in. The lesson here is that we should **NEVER** negate something without restricting ourselves first.

Definition (Universe)

Just like we restrict quantified statements to a "domain of individuals", we can (and if we want to take complements, **must**) restrict the elements of sets as well. We say $\mathscr{U}$ is the **universal set**. If we define such a $\mathscr{U}$, all sets are expected to only contain elements in $\mathscr{U}$.

Definition (Complement)

Let S be a set, and $\mathscr{U}$ be the universal set. Then,

$$\overline{S} := \mathscr{U} \setminus S$$

We want to prove that $S = \overline{\overline{S}}$.

$$
\begin{aligned}
S &= \{x \mid x \in S\} \\
&= \{x \mid \neg\neg(x \in S)\} && [p \iff \neg\neg p] \\
&= \{x \mid \neg(x \notin S)\} && [\text{Definition of } \notin] \\
&= \{x \mid \neg(x \in \overline{S})\} && [\text{Definition of } \overline{S}] \\
&= \{x \mid (x \notin \overline{S})\} && [\text{Definition of } \notin] \\
&= \{x \mid x \in \overline{\overline{S}}\} && [\text{Definition of } \overline{\overline{S}}] \\
&= \overline{\overline{S}}
\end{aligned}
$$

It follows that $S = \overline{\overline{S}}$.

(Again, if we did not have a universal set–this whole proof would be garbage.)

Let $\text{Days} = \{\text{M}, \text{W}, \text{F}\}$. Suppose we wanted to know the possible ways that we could allocate class days to be workshops. Let's call this set $\mathscr{P}(\text{Days})$:

$$\mathscr{P}(\text{Days}) = \{\varnothing, \{\text{M}\}, \{\text{W}\}, \{\text{F}\}, \{\text{M}, \text{W}\}, \{\text{W}, \text{F}\}, \{\text{M}, \text{F}\}, \{\text{M}, \text{W}, \text{F}\}\}$$

Definition (Powerset)

Let S be a set.

$$\mathscr{P}(S) := \{X \mid X \subseteq S\}$$

One last definition:

Definition (Cardinality)

Let S be a finite set. Then, $|S|$ ("the cardinality of S") is the number of elements in S.

Example (Cardinality)

- $|\varnothing| = 0$
- $|\{\varnothing\}| = 1$
- $|[n]| = n$

Sumtimes. . .

$$\vee$$

Again and Again